Examlet 1 **Foundations of Advanced Math** **2/3/23**

1. The sum of any two odd integers is even.

$2a+1$, $2b+1$

Suppose p and q are odd and are shown as $p = 2a+1$, $q = 2b+1$ where $a, b \in \mathbb{Z}$. When we add p and q, we get $p + q = 2a+1+2b+1 = 2a+2b+2 = 2(a+b+1)$. Since the definition of an even number is 2 times an integer and $a+b+1$ is an integer by closure, the sum of two odd integers is even □

Great!

2. $P \Rightarrow Q$ is logically equivalent to its contrapositive.

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$ *	$\neg Q \Rightarrow \neg P$ *
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The contrapositive of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$. As we can see in the truth table above, these two statements give the same results for all conditions of P and Q. Therefore, they are logically equivalent. □

Great!

3. If p, q, and r are integers for which $p|(q+r)$ and $p|q$, then $p|r$.

If $p|q+r$ then $q+r=p\cdot a$ for some $a \in \mathbb{Z}$
and if $p|q$ then $q=p\cdot b$ for some $b \in \mathbb{Z}$
Substituting we get $pb+r=pa$. Rearranged:
$r=pa-pb$ or $r=p(a-b)$. We see
$a-b$ is an integer by closure. Therefore
$p|r$. $\square$

Great!

4. $\sqrt{2}$ is irrational.

Suppose that $\sqrt{2}$ is rational so it can be written as $\frac{p}{q}$ for $p, q \in \mathbb{Z}, q \neq 0$, and reduced so p and q share no common factors. $\frac{p}{q}=\sqrt{2}$ gives $\frac{p^2}{q^2}=2$ which gives $p^2=2q^2$. We know that if $p \in \mathbb{Z}$ s.t. p^2 is even, then p itself must be even so we can write p as $2r$ for some $r \in \mathbb{Z}$. Substituting that in we have $(2r)^2=2q^2$ or $4r^2=2q^2$ or $2r^2=q^2$. But as previously stated, if $q \in \mathbb{Z}$ st. q^2 is even, then q itself must be even, meaning q and p share 2 as a common factor. Since this contradicts our supposition, we can conclude that $\sqrt{2}$ is irrational. $\square$

Great

5. Let S be a collection of n integers with the property that $\forall a \in S, a \equiv_5 1$. Let p be the product of all the integers in S. Then $p \equiv_5 1$.

First note that for $n=1$ and $n=0$ the statement is a little or a lot weird. The product of one number isn't the typical way to think of things, and the product of zero numbers is worse. But if you go with the pattern of x^1 and x^0 in the usual way it's all okay.

Now for real, let's induct, with $n=2$ as our base case. So we have a set with a and b where $a = 5r+1$ for some $r \in \mathbb{Z}$ and $b = 5s+1$ for some $s \in \mathbb{Z}$. Then $a \cdot b = (5r+1)(5s+1)$

$$= 25rs + 5r + 5s + 1$$
$$= 5(5rs + r + s) + 1$$

Since by closure $5rs + r + s$ is in $\mathbb{Z}$, we see $a \cdot b \equiv_5 1$ as desired.

Then s'pose the product of any k such integers is congruent modulo 5 to 1 and consider a set of $k+1$ such integers. The product p of the first k of them satisfies $p \equiv_5 1$ by inductive hypothesis. The product of that p with the $k+1^{st}$ element, call it t, must satisfy $p \cdot t \equiv_5 1$ by our base case. Then since the statement is true for $n=2$ and any time it's true of k it will also hold for $k+1$, it's true of all $n \geq 2$ by induction.