Examlet 2 **Foundations of Advanced Math** **2/24/23**

1. Write each of the following in as simple a way as possible:

(a) $[1,3]-[2,5] = [1,2)$

(b) $[1,3]\cap[2,5] = [2,3]$

(c) $[1,3]\cup[2,5] = [1,5]$

(d) $(3,6)-(6,8) = (3,6)$

(e) $(3,6)\cap(6,8) = \emptyset$

(f) $(3,6)\cup(6,8) = (3,6)\cup(6,8)$ or $(3,8)-\{6\}$

(g) $\bigcap_{n\in\mathbf{Z}^+}\left(\frac{-1}{n},\frac{1}{n}\right) = \{0\}$

(h) $\bigcup_{n\in\mathbf{Z}^+}\left(\frac{-1}{n},\frac{1}{n}\right) = (-1,1)$

Circle T or F for each of the following statements:

(i) $\{\emptyset\}\in\{\emptyset,a,b\}$ T F

(j) $\{\emptyset\}\subseteq\{\emptyset,a,b\}$ T F

$A = \{1\}$ $C = \{2\}$
$B = \{2, 3\}$

$\{2,3\}$ $\{1,2\}$

2. $(A \cup B) \cap C = A \cup (B \cap C)$

Counterexample.. Let $A = \{1\}$, $B = \{2,3\}$ and $C = \{2\}$.
Taking $(A \cup B) \cap C$ gives $(\{1\} \cup \{2,3\}) \cap \{2\}$ by substitution
By definition of union and intersection, this becomes
$\{1,2,3\} \cap \{2\}$ which becomes $\{2\}$.
Now taking $A \cup (B \cap C)$, substitution gives $\{1\} \cup (\{2,3\} \cap \{2\})$
which becomes $\{1\} \cup \{2\}$ which is equivalent to $\{1,2\}$.
The sets $\{2\}$ and $\{1,2\}$ are not equal, because $\{1,2\} \not\subseteq \{2\}$
therefore, $(A \cup B) \cap C \neq A \cup (B \cap C)$ by counterexample

Excellent

3. $A \cap \left(\bigcup_{i \in I} B_i \right)' = \bigcap_{i \in I} (A \cap B_i')$.

Take an $x \in A \cap \left(\bigcup_{i \in I} B_i \right)'$ so by definition $x \in A$ and $\neg[\exists i \in I, x \in B_i]$, this can be rewritten as $x \in A$ and $\forall i \in I, x \in B_i'$ by the box on page 12. This can also be rewritten as $\forall i \in I, x \in A$ and $x \in B_i'$ or $x \in \bigcap_{i \in I} (A \cap B_i')$ proving $A \cap \left(\bigcup_{i \in I} B_i \right)' \subseteq \bigcap_{i \in I} (A \cap B_i')$. Now take an $x \in \bigcap_{i \in I} (A \cap B_i')$ so by definition $\forall i \in I, x \in A$ and $x \in B_i'$ which can also be written as $x \in A$ and $\neg[\exists i \in I, x \in B_i]$ per the box on page 12. That also can be written as $x \in A \cap \left(\bigcup_{i \in I} B_i \right)'$ proving $\bigcap_{i \in I} (A \cap B_i') \subseteq A \cap \left(\bigcup_{i \in I} B_i \right)'$ $\therefore$ they are subsets of eachother and equal.

Great

4. (a) $\forall a, b, c \in \mathbb{R}, a < b \Rightarrow a - c < b - c$

Since $a < b$, the CAP lets us add $-c$ to both sides to get

$$a + (-c) < b + (-c)$$

or

$$a - c < b - c \quad . \ \square$$

(b) $\forall a, b, c \in \mathbb{R}, a < b$ and $c > 0 \Rightarrow \frac{a}{c} < \frac{b}{c}$

Lemma: If $c \in \mathbb{R}$ and $c > 0$, then $\frac{1}{c} > 0$.

Proof: Well, s'pose $\frac{1}{c} \leq 0$ for some $c > 0$. Then by CMP we can multiply by c to get $\frac{1}{c} \cdot c \leq 0 \cdot c$ or $1 \leq 0$, which is false, a contradiction arising from the supposition that $\frac{1}{c} \leq 0$, so it must be the case that $\frac{1}{c} > 0$. $\square$

Now, if $a < b$ and $c > 0$, it must be that $\frac{1}{c} > 0$, so by CMP

$$a \cdot \frac{1}{c} < b \cdot \frac{1}{c} \quad \text{or} \quad \frac{a}{c} < \frac{b}{c} \ . \ \square$$

5. $\forall x \in \mathbb{R}, -|x| \le x \le |x|$.

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Start with case that $x \ge 0$ so $|x| = x$. From this we know that $x \le |x|$. We know that $0 \ge -x$ and that $-|x| = -x$. So we know that $-|x| \le 0 \le x$ and by TPI, $-|x| \le x$ thus $-|x| \le x \le |x|$.

Now consider $x < 0$ so $|x| = -x$. We know that $-|x| = -(-x)$ or x so $-|x| \le x$. We know that $0 \le -x$ by CAP and $|x| = -x$. So we get $x < 0 < |x|$ or $x < |x|$ by TPI. Thus, $-|x| \le x \le |x|$.

Since the statement holds true for all possible values of x, it is true. $\square$

Nice!