1. Let $f, g : \mathbb{R} \to \mathbb{R}$.

(a) If f and g are both increasing, then $f + g$ is increasing.

If f and g are both increasing then by definition, $\forall x,y \in \mathbb{R} \ni x < y$, $f(x) \leq f(y)$ and $g(x) \leq g(y)$. Adding these inequalities we then get, $\forall x,y \in \mathbb{R} \ni x<y$, $f(x) + g(x) \leq f(y) + g(y)$, or in other terms, $f+g(x) \leq f+g(y)$. Since this is true for all $x,y \in \mathbb{R}$ where $x<y$, then $f+g$ is increasing by definition. $\square$

Excellent!

(b) If f and g are both increasing, then $f \cdot g$ is increasing.

Let $f(x) = x$ and $g(x) = x$. Then $f \cdot g(x) = f(x) \cdot g(x) = x \cdot x = x^2$. $f \cdot g(-1) = 1$ and $f \cdot g(0) = 0$. Since there exists an $x,y \in \mathbb{R}$ where $x<y$ but $f \cdot g(x)$ is not less than $f \cdot g(y)$, then $f \cdot g$ is not always necessarily increasing if both f and g are.

Nice!

2. If $f : A \to B$ and $g : B \to C$ are surjective functions, then $g \circ f$ is surjective.

Take any $c \in C$. Since g is surjective, we know $\exists\, b \in B$, $g(b) = c$. Now if we take this b, since f is surjective, we know $\exists\, a \in A$, $f(a) = b$. Therefore, $\forall c \in C$, $\exists a \in A$, $g \circ f(a) = c$. So $g \circ f$ is surjective. $\square$

Excellent!

3. If $f : A \to B$ is a bijection, then f is invertible.

Since f is bijective, it is also surjective so $\forall b \in B$, $\exists\ a \in A$, $f(a) = b$. Now, let us define $g: B \to A$ s.t $g(b) = a$, knowing that every pre-image $b \in B$ must have an image since f is surjective.

Now we must ensure that no $b \in B$ goes to more than one $a \in A$. This would mean that $a_1 = g(b) = a_2$ for two different $a_1, a_2 \in A$. This would also mean that $f(a_1) = b = f(a_2)$ by how we defined g, and since f is injective, this implies that $a_1 = a_2$.

Thus, we have constructed a function g, s.t. $\forall a \in A$, $g \circ f(a) = a$ and $\forall b \in B$, $f \circ g(b) = b$ so f is invertible. $\square$

Well done!

4. If A is equipollent to B, then B is equipollent to A.

If A is equipollent to B, then by definition, there exists a bijective function, f, mapping A to B. Because $f: A \to B$ is bijective, it is also invertible, meaning an inverse function, $g: B \to A$, exists. Because g is the inverse of f, we also know that f is the inverse of g, so by definition g is invertible. Because g is invertible, g is also bijective. Therefore, a bijective mapping from B to A exists, so by definition B is equipollent to A. Q.E.D.

Great

5. The set of throdd natural numbers is countable. (Yes, you need to include the details)

Well, let $f(n) = 3n+1$. This is a surjection since by definition every throdd integer is of the form $3n+1$ for some integer n, and to make it a natural $3n+1 \geq 0 \Rightarrow n \geq \frac{-1}{3}$, and the integers that are at lease $\frac{-1}{3}$ are exactly the naturals.

It's also an injection since if $f(n_1) = f(n_2)$ then $3n_1 + 1 = 3n_2 + 1$, so $3n_1 = 3n_2$ and $n_1 = n_2$.

So since $f(n) = 3n+1$ is both injective and surjective, it's bijective, and since a bijection exists from $\mathbb{N}$ to the set of throdd naturals, that set is by definition countable.