Examlet 4 **Foundations of Advanced Math** **4/14/23**

1. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b \Leftrightarrow 3|(a-b)$.

   (a) Determine whether and why $\sim$ is reflexive.

   $\sim$ is reflexive because $\forall a \in \mathbb{Z}$, $a-a=0$ and $3|0$ so $a \sim a$.

   (b) Determine whether and why $\sim$ is symmetric.

   Let $a \sim b$ so $a-b=3n$ for some $n \in \mathbb{Z}$. Then $-(a-b)=-3n$ or $b-a=3(-n)$ and since $-n$ is an integer under closure, $b \sim a$ so $a \sim b \Rightarrow b \sim a$ so $\sim$ is symmetric.

   (c) Determine whether and why $\sim$ is transitive.

   Let $a \sim b$ so $a-b=3n$ for some $n \in \mathbb{Z}$.
   Let $b \sim c$ so $b-c=3m$ for some $m \in \mathbb{Z}$
   We know $a=3n+b$ and $-c=3m-b$ so
   $a-c=(3n+b)+(3m-b)=3(n+m)$. Since $(n+m)$ is an integer under closure, we know that $a \sim b \wedge b \sim c \Rightarrow a \sim c$ so $\sim$ is transitive.

Excellent

2. Let $S = \{a, b, c, d, e\}$, and let $\sim = \{(a, a), (a, e), (b, b), (b, c), (c, b), (c, c), (d, d), (e, a), (e, e)\}$.

(a) Give the equivalence classes of $\sim$.

$[a] = \{a, e\}$

$[b] = \{b, c\}$

$[c] = \{c, b\}$

$[d] = \{d\}$

$[e] = \{e, a\}$

or

$[a] = [e] = \{a, e\}$

$[b] = [c] = \{b, c\}$

$[d] = \{d\}$

Great

(b) Give the partition associated with $\sim$.

$\{\{a, e\}, \{b, c\}, \{d\}\}$

3. Let S be a set and Π a partition of S. Let $\sim$ be a relation on S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.

(a) Show $\sim$ is a reflexive relation.

Take $a \in S$. Since Π is a partition of S, the union of the sets in Π contains all the elements in S. Therefore, $\exists P \in \Pi$ for which $a, a \in P$ meaning $a \sim a$. Thus $\sim$ is reflexive.

(b) Show $\sim$ is a symmetric relation.

If $a \sim b$ then $\exists P \in \Pi$ for which $a, b \in P$, meaning $\exists P \in \Pi$ for which $b, a \in P$. Therefore, $a \sim b \Rightarrow b \sim a$ so $\sim$ is symmetric.

(c) Show $\sim$ is a transitive relation.

S'pose $a \sim b$ and $b \sim c$. We'll say $a, b \in P_1$ and $b, c \in P_2$. But since all sets in Π are pairwise disjoint, b can not be in two different sets, meaning $P_1 = P_2$. Therefore, $a, b, c \in P_1$ so $a \sim b \wedge b \sim c \Rightarrow a \sim c$ and $\sim$ is transitive.

Good

4. Biff is a student at Enormous State University who has inexplicably found himself in a transition-to-proof course. Biff says "So we had to say if these things were, like, refluxive and symmetry and transitive, right? And one of them was if $a \sim b \Leftrightarrow a + b \equiv_5 0$, and I said it was refluxive because $5 \sim 5$, right? But they gave me no credit at all and then made fun of my answer in class the next day as an example of how not to ever do anything. I think I'm gonna drop."

Explain clearly to Biff what's wrong with his deduction.

Well, you found a single case that proves the relation is reflexive, this does not work to prove every number is reflexive. This is like me showing you a green apple and telling you all apples are green. By your proof I could say $6 \sim 6$ because $6+6 \equiv_5 0$ which just isn't true. In the future you could look for a counterexample before trying to prove something.

Nice!

Good

5. For two vertices v_1 and v_2 of a graph G we say $v_1 \sim v_2 \Leftrightarrow \exists$ a walk of even length from v_1 to v_2.

(a) Determine whether and why $\sim$ is reflexive.

$\forall v_0$ on G, $\exists$ a walk from v_0 to v_0 which is simply $\underline{v_0}$. Since this walk contains $\underline{0}$ edges, it is of $\underline{\text{even}}$ length, thus $v_0 \sim v_0$. Therefore, $\sim$ is reflexive.

(b) Determine whether and why $\sim$ is symmetric.

S'pose $v_1 \sim v_2$ so $\exists$ a walk of $\underline{\text{even}}$ length from v_1 to v_2. Then we know $\exists$ a walk of even length from v_2 to v_1 which is the walk from v_1 to v_2 but in the other $\underline{\text{direction}}$. Therefore, $v_1 \sim v_2 \Rightarrow v_2 \sim v_1$ so $\sim$ is symmetric.

(c) Determine whether and why $\sim$ is transitive.

S'pose $v_1 \sim v_2 \wedge v_2 \sim v_3$. We know $\exists$ a walk from v_1 to v_2 which contains $\underline{2n}$ edges for $n \in \mathbb{N}$. We also know that $\exists$ a walk from v_2 to v_3 containing $\underline{2m}$ edges for $m \in \mathbb{N}$. We know we can get a walk from v_1 to v_3 by $\underline{\text{combining}}$ the walk from v_1 to v_2 and the walk from v_2 to v_3 together. This would contain $2n+2m$ or $\underline{2(n+m)}$ edges, meaning $\exists$ a walk from v_1 to v_3 of even length. Therefore, $v_1 \sim v_2 \wedge v_2 \sim v_3 \Rightarrow v_1 \sim v_3$ so $\sim$ is transitive.

Great