Exam 2 **Calc 1** **10/4/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Great!

2. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	2	4	3	1	6	5
$g(x)$	5	3	2	6	1	4
$f'(x)$	3	7	4	2	6	5
$g'(x)$	4	1	3	8	10	3

(a) If $h(x) = f(x) \cdot g(x)$, what is $h'(3)$ and why?

$4 \times 2 + 3 \times 3$

$= 8 + 9$

$h'(3) = 17$

$f'(x) \times g(x) + f(x) \times g'(x)$

(b) If $h(x) = \frac{f(x)}{g(x)}$, what is $h'(2)$ and why?

$\frac{7 \times 3 - 4 \times 1}{3 \times 3} = \frac{21 - 4}{9} = \frac{17}{9}$

$h'(2) = \frac{17}{9}$

$\frac{f'(x) \times g(x) - f(x) \times g'(x)}{g(x) \times g(x)}$

(c) If $h(x) = f(g(x))$, what is $h'(5)$ and why?

$f'(1) \times 10$

$= 3 \times 10$

$h'(5) = 30$

$f'(g(x)) \times g'(x)$

(d) If $h(x) = g(f(x))$, what is $h'(5)$ and why?

$g'(6) \times 6$

$= 3 \times 6$

$h'(5) = 18$

$g'(f(x)) \times f'(x)$

Excellent

x_0 y_0

3. Write an equation for the line tangent to $f(x) = x^3$ at $(2, 8)$.

$f'(x) = 3x^2$

slope comes from derivative

plug in x

$3(2)^2$

$\boxed{= 12}$

$y - y_0 = m(x - x_0)$

$\boxed{y - 8 = 12(x - 2)}$

Excellent!

W

rewritten

4. Prove that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

sum rule

$(f+g)' = f'(x) + g'(x)$

plug into definition of derivative

$$\frac{d}{dx} = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h}$$

because f and g are differentiable

Excellent

$$= \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{f'(x)} + \lim_{h \to 0} \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)}$$

+

5. Find the derivative of $f(x) = \dfrac{\cos x}{\sin x}$.

$$f(x) = \frac{\cos x}{\sin x}$$

$$f'(x) = \frac{-\sin x \cdot \sin x - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-[\sin^2 x + \cos^2 x]}{\sin^2 x}$$

Excellent

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

#

6. State and prove the Quotient Rule for derivatives.

If f and g are differentiable functions then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

wherever $g(x) \neq 0$.

Proof: Well,

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{g(x+h)\cdot g(x)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) + f(x)\cdot g(x) - f(x)\cdot g(x+h)}{g(x+h)\cdot g(x)\cdot h}$$

$$= \lim_{h\to 0} \left[\underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{since } f \text{ is diff.}}\cdot\frac{g(x)}{g(x+h)\cdot g(x)} - \frac{f(x)}{g(x+h)\,g(x)}\cdot\overbrace{\frac{g(x+h)-g(x)}{h}}^{\text{since } g \text{ is diff.}}\right]$$

$$= \cancel{\lim_{h\to 0}}\ \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\underbrace{g(x)}_{\text{since } g \text{ is diff. and therefore cont.}}\cdot g(x)}$$

7. Sam is a calculus student at Enormous State University, and they're having some trouble with derivatives. Sam says "Wow, this derivative stuff is confusing. What really gets me is the product and quotient and chain stuff. So like, we're supposed to work in groups in class, which is sad because none of us know how to do it so we're just confused together. But one person said for sine squared of x we should do the chain rule, and this other person said we should write it as sine times sine and use the product rule. How do you tell which way is right?"

Help Sam by explaining, as clearly as possible, how to decide between their options.

Sam,

If you use product rule for → $f(x) = \sin^2 x$

$f'(x) = \sin x \ \sin x$

$= \sin x \cdot \cos x + \cos x \cdot \sin x$

$= 2\sin x \cos x$

& If you use chain rule, $f(x) = (\sin x)^2$

$= 2(\sin x)^{2-1} \cdot \cos x$

$= 2\sin x \cos x$

You would get the same result You can even use the power rule It's just that you should be able to differentiate & know the basic formula of derivative. You will figure out everything.

Good

8. (a) Find the linearization $L(x)$ of the function $f(x) = \sqrt{x}$ at $x = 4$.

$f'(x) = \frac{1}{2\sqrt{x}}$ $y - y_0 = m(x - x_0)$ $f(4) = 2$

$f'(4) = \frac{1}{2\sqrt{4}}$ $y = L(x)$ So our point is $(4,2)$

$L(x) - 2 = \frac{1}{2\sqrt{4}}(x-4)$

$L(x) = \frac{1}{2\sqrt{4}}(x-4) + 2$

$L(x) = \frac{1}{4}(x-4) + 2$

(b) Use your linearization from part a to approximate $\sqrt{4.1}$.

$L(4.1) = \frac{1}{4}(4.1-4) + 2$

$L(4.1) = \frac{1}{4}(0.1) + 2$

$L(4.1) = 0.025 + 2$

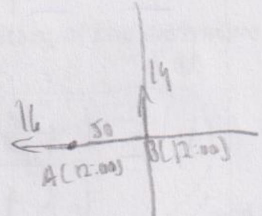
$L(4.1) = 2.025$

Excellent!

9. At noon, ship A is 50 nautical miles due west of ship B. Ship A is sailing west at 16 knots and ship B is sailing north at 19 knots. How fast (in knots) is the distance between the ships changing at 3 PM?

$[a^2+b^2=c^2]'$

$= 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$



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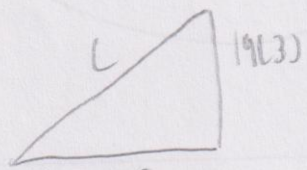
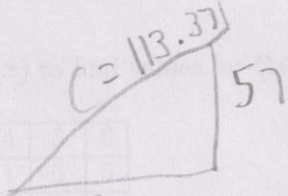




$$2(98)\cdot 16 + 2(57)\cdot 19 = 2(113.371)\cdot\frac{dc}{dt}$$

$$3136 + 2166 = 226.742\cdot\frac{dc}{dt}$$

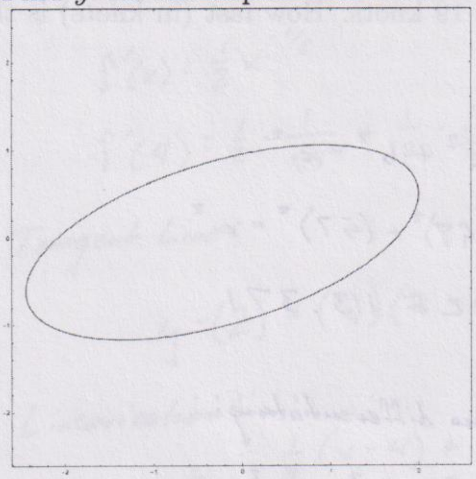
$$5302 = 226.742\cdot\frac{dc}{dt}$$

$$\frac{dc}{dt} = \frac{5302}{226.742}$$

Excellent!

$$\frac{dc}{dt} = 23.383404$$

10. (a) Find y' for the implicit curve $3x^2 - 6xy + x + 12y^2 = 11$.



$$6x - 6y - 6x \cdot y' + 1 + 24y \cdot y' = 0$$

$$24y \cdot y' - 6x \cdot y' = 6y - 6x - 1$$

$$y'(24y - 6x) = 6y - 6x - 1$$

$$y' = \frac{6y - 6x - 1}{24y - 6x}$$

For $-6xy$:

$$-6 \cdot y + -6x \cdot 1 \cdot y'$$

(b) What is the slope of the tangent line to the curve from part a at the point $(2, \frac{1}{2})$?

At $(2, \frac{1}{2})$, $y' = \dfrac{6(\frac{1}{2}) - 6(2) - 1}{24(\frac{1}{2}) - 6(2)} = \dfrac{3 - 13}{12 - 12}$

Infinite! Vertical!