Exam 3 **Calc 1** **10/23/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. For each for the following functions, give $f'(x)$. No justification required this time.

   (a) $f(x) = \ln x$

   $f'(x) = \frac{1}{x}$

   (b) $f(x) = \arcsin x$

   $f'(x) = \frac{1}{\sqrt{1-x^2}}$

   Good

   (c) $f(x) = \arctan x$

   $f'(x) = \frac{1}{1+x^2}$

2. Find an equation for the line tangent to $f(x) = \ln(x)$ at $x = 2$.

$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2}$

$y - \ln(2) = \frac{1}{2}(x-2)$

Great

3. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	2	4	3	1	6	5
$g(x)$	5	3	2	6	1	4
$f'(x)$	3	7	4	2	6	5
$g'(x)$	4	1	3	8	10	3

(a) If $h(x) = \ln(f(x))$, what is $h'(3)$ and why? chain rule $f'(g(x)) \cdot g'$

$h'(x) \quad \frac{1}{f(x)} \cdot f'(x)$

$h'(3) = \frac{1}{f(3)} \cdot f'(3)$

$h'(3) = \frac{1}{3} \cdot 4 = \boxed{\frac{4}{3}}$

(b) If $h(x) = \underbrace{g(x)}_{f} \cdot \underbrace{\arctan x}_{g}$, what is $h'(4)$ and why? Product Rule $f' \cdot g + f \cdot g'$

$h'(x) = g'(x) \cdot \arctan x + g(x) \cdot \frac{1}{1+x^2}$

$h'(4) = g'(4) \cdot \arctan(4) + g(4) \cdot \frac{1}{1+(4)^2}$

Great

$\boxed{h'(4) = 8 \cdot \arctan(4) + 6 \cdot \frac{1}{1+(4)^2}}$

4. Why is the derivative of $\ln x$ equal to $\frac{1}{x}$?

I know $\ln x$ and e^x are inverses, so,

$$e^{\ln x} = x$$

differentiating, (chain rule!)

$$e^{\ln x} \cdot (\ln x)' = 1$$

solve for $(\ln x)'$,

$$(\ln x)' = \frac{1}{e^{\ln x}}$$

Excellent!

remember that $e^{\ln x} = x$!

$$(\ln x)' = \frac{1}{x}$$

5. Evaluate $\lim_{x \to \infty} \frac{3x^2}{e^x}$

$\lim_{x \to \infty} \frac{3x^2}{e^x}$ is in $\frac{\infty}{\infty}$ form.

Using L'Hopitals rule,

$\overset{L'H}{=} \lim_{x \to \infty} \frac{6x}{e^x}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{6}{e^x}$

~~lim~~ $= 0$ #

Great

6. Why is the derivative of arcsin x equal to $\frac{1}{\sqrt{1-x^2}}$?

I know,

$\sin(\arcsin(x)) = x$

differentiating, (chain rule!)

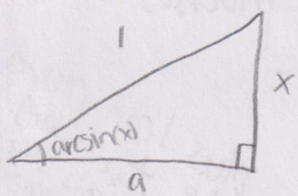
$\cos(\arcsin(x)) \cdot (\arcsin(x))' = 1$

Solve for $(\arcsin(x))'$;

$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$

Excellent!

can't leave the bottom like that → TRIGTIME



$S\frac{O}{H}$ $C\frac{A}{H}$ $T\frac{O}{A}$

→ find missing side

$a^2 + x^2 = 1^2$

$\sqrt{a^2} = \sqrt{1-x^2}$

$a = \sqrt{1-x^2}$

→ now find cos(arcsin(x))

$\cos(\arcsin(x)) = \frac{\sqrt{1-x^2}}{1}$ or $\sqrt{1-x^2}$

→ now plug back into $(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$

$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! Calculus is hard enough, but there's this really stupid guy in my class who makes it even more impossible! We were supposed to work in pairs on this one problem, and the guy said that $\tan^{-1} x$ could break up into $\dfrac{\sin^{-1} x}{\cos^{-1} x}$ and then we could do this one problem. But I don't even think that's right, is it?"

Help Bunny understand whether the suggested expression is equivalent.

Bunny, it sounds like that guy is trying to avoid thinking instead of getting better at it. When you're faced with something unfamiliar like that, it's often good to work from what you do know. It's true that $\tan x = \frac{\sin x}{\cos x}$, but this isn't the same as that, so try something. If you put "1" in for x in the proposed equation, you have $\tan^{-1} 1$, which means what angle has tangent of 1, which is $\pi/4$ (you can get help from a calculator). But then $\sin^{-1} 1 = \pi/2$ since $\sin \pi/2 = 1$, ~~and~~ $\cos^{-1} 1 = 0$ since $\cos 0 = 1$. But then $\frac{\sin^{-1} 1}{\cos^{-1} 1} = \frac{\pi/2}{0}$, and that's definitely not equal to $\pi/4$.

So also remember that just working for a couple of values wouldn't guaranty it always works - if you tried putting 0 in they would have matched! But if they don't match, spot-checking like this usually makes it clear pretty easily.

8. A particular cell phone battery has a peak capacity of 5500mAh at the time of purchase, and after 8 months has a peak capacity of 92% of the original.

(a) Find a function of the form $f(x) = A\,b^x$ for the peak capacity of the battery after x months have passed.

$.92 \cdot 5500$

Time purchase $(0, 5500)$

After 8 months $(8, 5060)$

$f(x) = ab^x$

$5060 = (5500)(b)^8$

$5500 = ab^0$

$\frac{5060}{5500} = b^8$

$5500 = a\,1$

$f(x) = (5500)\left(\sqrt[8]{\frac{5060}{5500}}\right)^x$ Good

$5500 = a$

$\sqrt[8]{\frac{5060}{5500}} = b$

(b) What does your formula project the peak capacity will be after 12 months?

$f(x) = (5500)\left(\sqrt[8]{\frac{5060}{5500}}\right)^{12}$

$f(x) = 4853.381502$

(c) At what rate does your formula predict the peak capacity will be changing after 12 months?

12 months

$f'(x) = (5500)\left(\sqrt[8]{\frac{5060}{5500}}\right)^x \cdot \ln\left(\sqrt[8]{\frac{5060}{5500}}\right)$

$(5500)(.989631427)^{12} \cdot \ln(.989631427)$

$a^x = a^x \cdot \ln a$

$f(x) = 4853.381501 \cdot -.010422701$

Great

$f(x) = -50.58534422$

Meaning peak capacity will keep decreasing

9. If $f(x) = \frac{1}{a} \arctan \frac{x}{a}$, what is $f'(x)$?

Chain Rule!

$$f'(x) = \frac{1}{a} \cdot \frac{1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1}{a}$$

$$= \frac{1}{a^2} \cdot \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{a^2}{a^2}$$

$$= \frac{1}{a^2} \cdot \frac{a^2}{a^2+x^2}$$

$$= \frac{1}{a^2+x^2}$$

10. Can you use the fact that the derivative of a^x is $(\ln a)a^x$ to find the derivative of $\log_a x$?

Well I know: $a^{\log_a x} = x$

Differentiating: $(\ln a)\, a^{\log_a x} \cdot (\log_a x)' = 1$

Solving for $(\log_a x)'$: $(\log_a x)' = \frac{1}{(\ln a)\, a^{\log_a x}}$

But we know $a^{\log_a x} = x \ldots$

$$(\log_a x)' = \frac{1}{(\ln a)\, x}$$