Exam 1 **Calc 3** **9/27/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Great!

2. Show that the function $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ fails to have a limit at $(0, 0)$.

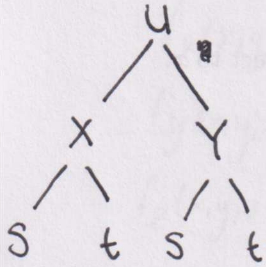
along $y=x$ $\quad \lim_{(x,x)\to(0,0)} \dfrac{2x(x)}{x^2+x^2} = \lim_{(x,x)\to 0} \dfrac{2x^2}{2x^2} = 1$

along y axis, $x=0$, $\quad \lim_{(0,y)\to(0,0)} \dfrac{2(0)y}{0^2+y^2} = 0$

Since the limit along two different directions are not equal, therefore the limit of $f(x,y)$ does not exist.

Great

3. Suppose that $u = f(x, y)$, where $x = x(s,t), y = y(s,t)$. Write the Chain Rule formula for $\frac{\partial u}{\partial t}$. Make very clear which derivatives are partials.



$$\frac{\partial u}{\partial t} = \underbrace{\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}}_{\text{all partials!}}$$

Great

4. Find the directional derivative of $f(x, y) = x^3 - xy + 2x$ at the point $(-2, 2)$ in the direction of the vector $\langle -3, 4 \rangle$.

$D_{\vec{u}}(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$

$f_x(x,y) = 3x^2 - y + 2 \Rightarrow f_x(-2,2) = 3(-2)^2 - 2 + 2 = 12$

$f_y(x,y) = -x \Rightarrow f_y(-2,2) = -(-2) = 2$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$, where $|\vec{v}| = \sqrt{-3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

$\vec{u} = \langle \frac{-3}{5}, \frac{4}{5} \rangle$

Excellent!

$D_{\vec{u}}(-2,2) = \frac{-3}{5}(12) + \frac{4}{5}(2)$

$= \frac{-36}{5} + \frac{8}{5} = \boxed{\frac{-28}{5}}$

$$\frac{u'v - v'u}{v^2}$$

5. Let $f(x, y) = \dfrac{x}{3x + 2y}$. In which direction is the directional derivative greatest at the point ~~(2, −3)~~? $(-2, 4)$

The direction at which the directional derivative is greatest is the same as the direction of the gradient which is the direction of steepest ascent

$$\frac{\partial f}{\partial x} = \frac{1(3x+2y) - (3)(x)}{(3x+2y)^2} = \frac{2y}{(3x+2y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{0(3x+2y) - (2)(x)}{(3x+2y)^2} = \frac{-2x}{(3x+2y)^2}$$

$$\nabla f = \left\langle \frac{2y}{(3x+2y)^2}, \frac{-2x}{(3x+2y)^2} \right\rangle$$

Excellent!

$$\nabla f(-2, 4) = \langle 2, 1 \rangle$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$ the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

evaluating by minors

well, let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Then $\vec{a} \times \vec{b} =$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{array} = \vec{i}(a_2 b_3 - a_3 b_2) - \vec{j}(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

$= \langle a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1 \rangle$

Now, since we know the most important thing about dot product: when a vector is dotted with another vector and the result is zero, those two vectors are $\perp$.

so we can dot $\vec{a}$ with $\vec{a} \times \vec{b}$:

$\langle a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$

$= a_1(a_2 b_3 - a_3 b_2) + a_2(-a_1 b_3 + a_3 b_1) + a_3(a_1 b_2 - a_2 b_1)$

$= a_1 \cancel{a_2} b_3 - a_1 \cancel{a_3} b_2 - a_1 \cancel{a_2} b_3 + a_2 \cancel{a_3} b_1 + a_1 \cancel{a_3} b_2 - a_2 \cancel{a_3} b_1$

$= \underline{0}$

$\therefore \vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$.

Excellent! ☺

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. The professor told us there were these things we definitely had to know for the test, like in my notes I have that she said that the level curvy things are ninety degrees from the direction of greatest increase. And she said we have to know why that's true, but I totally don't have a clue. I looked in the book and it makes no sense at all. She never said anything about it in class, just during the review. So how am I supposed to know why it's true? This is so unfair!"

Explain clearly to Bunny how she could deduce such a conclusion from other things which she should indeed know.

We define level curves as places where the directional derivative is zero. On a level curve, we know that the gradient points in the direction where the directional derivative is greatest, meaning the directional derivative, $D_{\vec{u}} = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f||\vec{u}|\cos\theta$, is at a maximum. If $|\vec{\nabla} f||\vec{u}|\cos\theta$ is at a maximum, that means $\cos\theta = 1$ and thus $\theta = 0$. This makes sense, because the angle being zero tells us that our vector is pointing in the direction of the gradient. Now turn left or right, so $\theta = 90°$ or $-90°$, and in either case $\cos\theta = 0$, meaning the directional derivative is zero, which is precisely how we defined a level curve. From this, we can conclude that level curves are perpendicular to the gradient, or the direction where the directional derivative is greatest.

Great!

8. Find and classify all critical points of $f(x, y) = x^2y + y^3 - 75y$.

$f_x = 2xy$

$f_y = x^2 + 3y^2 - 75$

$2xy = 0 \Rightarrow x = 0$ or $y = 0$

$x^2 + 3y^2 - 75 = 0$

Crit Points:

$(0,5)$: Minimum

$(0,-5)$: Maximum

$(\sqrt{75}, 0)$: Saddle

$(-\sqrt{75}, 0)$: Saddle

If $x=0$:

$3y^2 - 75 = 0$

$3(y^2 - 25) = 0$

$y = 5$ or $y = -5$

If $y=0$:

$x^2 - 75 = 0$

$x^2 = 75$

$x = \pm\sqrt{75}$

$(0,5), (0,-5)$

$f_{xx} = 2y$

$f_{yy} = 6y$

$f_{xy} = 2x$

Nice Job!

Let's Classify!

$D(0,5) = (10)(30) - (0)^2$

$= > 0$ and $f_{xx} > 0 \Rightarrow$ min

$D(0,-5) = (-10)(-30) - (0)^2$

$= > 0$ and $f_{xx} < 0 \Rightarrow$ max

$D(\sqrt{75}, 0) = (0)(0) - (\sqrt{75})^2$

$= < 0 \Rightarrow$ saddle

$D(-\sqrt{75}, 0) = (0)(0) - (-\sqrt{75})^2$

$= < 0 \Rightarrow$ saddle

9. Find all extrema of the function $f(x,y) = 2x + 2y + z$ subject to the constraint $x^2 + y^2 + z^2 = 4$.

$$\nabla f = \lambda \nabla g$$

$$\langle 2, 2, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$2 = 2x\lambda \Rightarrow x = \frac{1}{\lambda}$$
$$2 = 2y\lambda \Rightarrow y = \frac{1}{\lambda} \qquad \Rightarrow x = y$$
$$1 = 2z\lambda \Rightarrow z = \frac{1}{2\lambda}$$

$x^2 + y^2 + z^2 = 4$ becomes $\frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 4$

$$\frac{9}{16} = \lambda^2$$
$$\lambda = \pm \frac{3}{4}$$

Plugging back in for x, y, and z we get

$\left(\frac{4}{3}, \frac{4}{3}, \frac{2}{3}\right)$ and $\left(\frac{-4}{3}, \frac{-4}{3}, \frac{-2}{3}\right)$ as the points

and

$\frac{8}{3} + \frac{8}{3} + \frac{2}{3} = \frac{18}{3} = 6$ and -6 the extreme values.

$2x = \lambda 3x$
$-2z = \lambda z$
$\lambda = -2$
$2y = \lambda y$
$\lambda = 2$

10. Find the points on the surface $x^2 + y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $3x + y + z = 1$.

tangent of $x^2+y^2-z^2 \Rightarrow \nabla f(x) = \langle 2x, 2y, -2z \rangle$

$\nabla f(x)$ is parallel to $3x+y+z=1$

$\nabla f(x) = \lambda \nabla g(x)$

$\langle 2x, 2y, -2z \rangle = \lambda \langle 3, 1, 1 \rangle$

$2x = 3\lambda$ $\quad x = \frac{3}{2}\lambda$

$2y = \lambda$ $\quad y = \frac{\lambda}{2}$

$-2z = \lambda$ $\quad z = -\frac{\lambda}{2}$

$\left(\frac{3}{2}\lambda\right)^2 + \left(\frac{\lambda}{2}\right)^2 - \left(\frac{-\lambda}{2}\right)^2 = 1$

$\frac{9}{4}\lambda^2 + \cancel{\frac{\lambda^2}{4}} - \cancel{\frac{\lambda^2}{4}} = 1$

$\frac{9}{4}\lambda^2 = 1$

$\lambda^2 = \frac{4}{9}$ $\quad \lambda = \pm \frac{2}{3}$

for $\lambda = \frac{2}{3}$

$x = 1$

$y = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

$z = -\frac{2}{3} \times \frac{1}{2} = -\frac{1}{3}$

for $\lambda = -\frac{2}{3}$

$x = -1$

$y = -\frac{1}{3}$

$z = \frac{1}{3}$

Excellent

points where they are parallel

$\Rightarrow (1, \frac{1}{3}, -\frac{1}{3})$ & $(-1, -\frac{1}{3}, \frac{1}{3})$

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