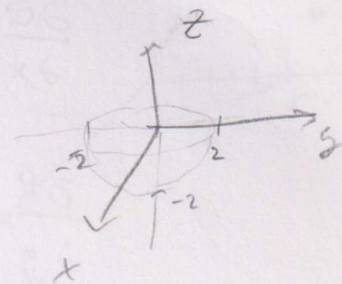
Exam 3 **Calc 3** **11/22/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Parametrize and give bounds for the bottom half of a sphere with radius 2 centered at the origin.



$x(u,v) = 2\sin u \cos v$

$y(u,v) = 2\sin u \sin v$

$z(u,v) = 2\cos u$

Excellent!

$\frac{\pi}{2} \le u \le \pi$

$0 \le v \le 2\pi$

2. Let **F** be the vector field $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + x^2\mathbf{j}$. Let C be the line segment from (2,1) to the origin. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\vec{F} = \langle 2xy, x^2 \rangle$ line segment $\longrightarrow$ PF? yes $\longrightarrow$ FTLI

$f_x = 2xy$ $\quad f_{xy} = 2x$

$f_y = x^2$ $\quad f_{yx} = 2x$) equal

$f(x,y) = x^2 y$ ← potential fun.

$$\int_C \vec{F} \cdot d\vec{r} = f(x,y)\Big|_{(2,1)}^{(0,0)}$$

$= f(0,0) - f(2,1)$

$= 0 - (2^2(1))$

$= -4(1)$

$\boxed{= -4}$

Great

$$\overset{P}{\overbrace{y^3\,dx}} - \overset{Q}{\overbrace{x^3\,dy}}$$

3. Evaluate $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2+y^2=4$, oriented counterclockwise.

Closed path $\rightarrow$ Green's Thm!

$$\int_C y^3 dx - x^3 dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \frac{\partial Q}{\partial x} = -3x^2, \quad \frac{\partial P}{\partial y} = 3y^2$$

Excellent!

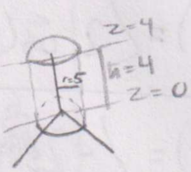
$$= \iint (-3x^2 - 3y^2)\, dA = \iint -3(\underbrace{x^2+y^2}_{r^2})\, dA$$

Add Jacobian

$$= \int_0^{2\pi}\int_0^2 -3r^3\, dr\, d\theta = \int_0^{2\pi} \left.\frac{-3r^4}{4}\right|_0^2 d\theta = \int_0^{2\pi} -12\, d\theta = \boxed{-24\pi}$$

4. Let $\mathbf{F}(x,y,z) = \langle 2x, 3z, -3z\rangle$, and let S be the cylinder $25 = x^2+y^2$ between $z=0$ and $z=4$, along with a disc of radius 5 in the plane $z=0$ centered at $(0,0,0)$ and another disc of radius 5 in the plane $z=4$ centered at $(0,0,4)$, all with outward orientation. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

Divergence Theorem: Closed surface

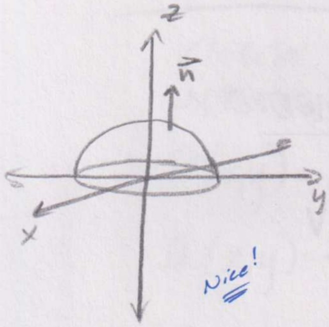


$$\nabla\cdot\vec{F} = 2 + 0 - 3 = -1$$

$$\Rightarrow -1\left(\begin{array}{c}\text{volume of a cylinder} \\ \text{w/ } r=5 \text{ and } h=4\end{array}\right) = -1\left(\pi(5)^2\cdot 4\right) = -100\pi$$

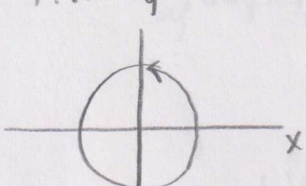
Excellent!

5. Let $\mathbf{G}(x, y, z) = \langle 5x, -z, y \rangle$. Let S be the portion of $x^2 + y^2 + z^2 = 4$ above $z = 0$, with upward orientation. Evaluate $\iint_S \text{curl } \mathbf{G} \cdot d\mathbf{S}$.



Nice!

T.V.



* counter clockwise

curl $\Rightarrow$ Stoke's thm

$\iint_S \text{curl}\, \vec{G} \cdot d\vec{S} = \int_C \vec{G} \cdot d\vec{r}$

I $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle, \quad 0 \le t \le 2\pi$

II $\vec{G}(\vec{r}(t)) = \langle 10\cos t, 0, 2\sin t \rangle$

III $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

IV $\int_0^{2\pi} \langle 10\cos t, 0, 2\sin t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$

V $\int_0^{2\pi} -20 \sin t \cos t \, dt$

$\rightarrow$ u-sub: $u = \cos t$
$du = -\sin t \, dt$

Excellent!

$= 20 \left[\frac{\cos^2 t}{2} \right]_0^{2\pi}$

$= 20 \left[\frac{1}{2} - \frac{1}{2} \right] = \underline{\underline{0}}$

6. Show that for any scalar function $f(x, y, z)$ with continuous second-order partial derivatives, $\text{curl}(\nabla f) = \mathbf{0}$. Make it clear how the requirement that the partials be continuous is important.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\text{Curl}(\nabla f) = \nabla \times \langle f_x, f_y, f_z \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

Now, by Clairaut's Theorem, since f has continuous second-order partial derivatives, we are guaranteed that

$$f_{zy} = f_{yz}, \quad f_{xz} = f_{zx}, \text{ and } f_{yx} = f_{xy}$$

$$\therefore \text{curl}(\nabla f) = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle = \vec{0}$$

Great!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. This stuff is so totally confusing. I thought math was where you, like, did problems and stuff, but now there's all this vocabulary stuff, and it's more like French than math or something, you know? So our professor was saying that we should understand why line integrals on <u>closed paths</u> in <u>conservative</u> vector fields are always zero, and I'm not even sure what those words are. Help!"

Explain clearly to Bunny what the underlined terms mean and why the conclusion is valid.

A conservative vector field $\vec{F}$ has a potential function f, such that the gradient of f is equal to $\vec{F}$, $\vec{F} = \nabla f$.

When there is a conservative vector field, we can use the Fundamental Theorem for Line Integrals, $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$.

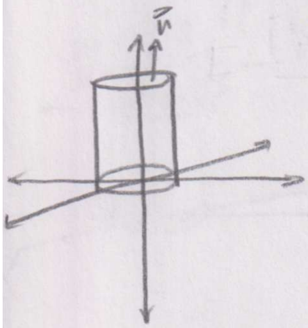
This means you can plug the endpoint, b, the potential function and then subtract the startpoint, a, plugged into the potential function.

When a path is closed, the startpoint and the end point are exactly the same. Therefore, when you do the subtraction, it will be 0.

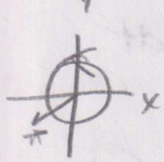
Wonderful!

Long Way

8. Let $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$ and let S be the portion of the cylinder $x^2+y^2=9$ between $z=0$ and $z=2$ with outward orientation. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$.



T.V.



I $\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v\rangle$

$0 \le u \le 2\pi$

$0 \le v \le 2$

II $\vec{F}(\vec{r}(u,v)) = \langle -3\sin u, 3\cos u, 0\rangle$

III $\vec{r}_u(u,v) = \langle -3\sin u, 3\cos u, 0\rangle$

$\vec{r}_v(u,v) = \langle 0, 0, 1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \hat{\imath}(3\cos u) - \hat{\jmath}(-3\sin u) + \hat{k}(0)$$

IV $\displaystyle\int_0^{2\pi}\int_0^2 \langle -3\sin u, 3\cos u, 0\rangle \cdot \langle 3\cos u, 3\sin u, 0\rangle \, dv\,du$

V $\displaystyle\int_0^{2\pi}\int_0^2 -9\sin u\cos u + 9\cos u\sin u \, dv\,du$

Great!

$$= \int_0^{2\pi}\int_0^2 0\,dv\,du = \underline{\underline{0}}$$

9. Let $\mathbf{F}(x,y) = Q(x,y)\mathbf{j}$. Let C be the line segment from $(\overset{2}{c},b)$ to $(\overset{9}{d},b)$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\vec{r} = \langle c+(d-c)t,\ b\rangle$$
$$\vec{r}\,' = \langle d-c,\ 0\rangle$$

$$\int F\cdot dr = \int_0^1 \langle 0,Q\rangle \cdot \langle d-c,0\rangle\, dt$$

$$= \int_0^1 (0)(d-c) + Q(x,y)\cdot(0)\ dt = \int_0^1 0 = 0$$

Good

10. Let $\mathbf{G}(x,y,z) = \langle 2, z, -y \rangle$ and let S be the portion of the plane $z = 1$ in the first quadrant inside the cylinder $x^2 + y^2 = 9$, with upward orientation. Evaluate $\iint_S \mathbf{G} \cdot d\mathbf{S}$.

I. $\vec{r}(u,v) = \langle u\cos v, u\sin v, 1 \rangle$ for $0 \le u \le 3$, $0 \le v \le \frac{\pi}{2}$

II. $\vec{G}(\vec{r}(u,v)) = \langle 2, 1, -u\sin v \rangle$

III. $r_u = \langle \cos v, \sin v, 0 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \langle 0, 0, u\cos^2 v - -u\sin^2 v \rangle$$

$$= \langle 0, 0, u(\cos^2 v + \sin^2 v) \rangle$$

$$= \langle 0, 0, u \rangle$$

IV. $\iint_S \vec{G} \cdot d\vec{S} = \iint \langle 2, 1, -u\sin v \rangle \cdot \langle 0, 0, u \rangle \, dA$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 -u^2 \sin v \, du \, dv$$

$$= \int_0^{\frac{\pi}{2}} -\frac{u^3}{3} \sin v \Big|_0^3 \, dv$$

$$= \int_0^{\frac{\pi}{2}} -9\sin v \, dv$$

$$= 9\cos v \Big|_0^{\pi/2}$$

$$= 9 \cdot \cos\frac{\pi}{2} - 9\cos 0$$

$$= 0 - 9$$

$$= -9$$