Exam 1 **Real Analysis 1** **9/25/24**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of a sequence $\{a_n\}$ converging.

A sequence $\{a_n\}$ converges to a limit L iff $\forall \varepsilon > 0 \ \exists n^* \ni n > n^* \Rightarrow |a_n - L| < \varepsilon$

Great

2. State the definition of an accumulation point.

A set S has an accumulation point s_0 iff $\forall \varepsilon > 0 \ \exists t \in S \ni 0 < |t - s_0| < \varepsilon$

Great

3. a) State the definition of a Cauchy sequence.

A sequence is Cauchy iff $\forall \varepsilon > 0\ \exists n^*$ such that $n, m > n^*$ implies $|a_n - a_m| < \varepsilon$.

b) State the Cauchy Convergence Criterion.

Any sequence in $\mathbb{R}$ is Cauchy if and only if it converges.

4. Give an example of a sequence of positive numbers which is decreasing but not convergent, or say why it's not possible.

Well, since this sequence is monotone and bounded, then it must converge by the Monotone Convergence Theorem. So this is not possible.

Excellent

5. Prove directly from the definition that $\lim_{x\to\infty} \frac{x}{x+1} = 1$.

Let $\varepsilon > 0$ be given. Take $M = \frac{1}{\varepsilon}$.

Then $x > \frac{1}{\varepsilon}$ certainly implies

$x > \frac{1}{\varepsilon} - 1$ which we rearrange as

$x + 1 > \frac{1}{\varepsilon}$ or

$\frac{1}{x+1} < \varepsilon$ Since $\varepsilon > 0$ we know $M > 0$ and $x > 0$ and $\frac{1}{x+1} > 0$, so

$$\left|\frac{-1}{x+1}\right| = \left|\frac{1}{x+1}\right| < \varepsilon \quad \text{or}$$

$$\left|\frac{x - x - 1}{x+1}\right| < \varepsilon \quad \text{or}$$

$$\left|\frac{x}{x+1} - \frac{x+1}{x+1}\right| < \varepsilon \quad \text{or}$$

$$\left|\frac{x}{x+1} - 1\right| < \varepsilon.$$

So since $x > M$ implies $|f(x) - 1| < \varepsilon$, we conclude that by definition

$$\lim_{x\to\infty} \frac{x}{x+1} = 1. \quad \square$$

Don't look over here!

I want:

$$\left|\frac{x}{x+1} - 1\right| < \varepsilon$$

$$\left|\frac{x}{x+1} - \frac{x+1}{x+1}\right| < \varepsilon$$

$$\left|\frac{x - x - 1}{x+1}\right| < \varepsilon$$

$$\left|\frac{-1}{x+1}\right| < \varepsilon$$

$$\frac{1}{x+1} < \varepsilon$$

$$\frac{1}{\varepsilon} < x + 1$$

$$\frac{1}{\varepsilon} - 1 < x$$

$$\frac{1}{\varepsilon} < x$$

6. Suppose that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, where f and g are functions with domain D. Prove (directly from the definition) that $\lim_{x \to a}[f(x) \cdot g(x)] = A \cdot B$.

Let $\varepsilon > 0$ be given.

We know that there exits some $G \in \mathbb{R}$ such that $|g(x)| < G$ for all $x \in D$ since $g(x)$ is convergent.

Since $\lim_{x \to a} f(x) = A$, then $\exists \delta_1 > 0$ such that $|f(x) - A| < \frac{\varepsilon}{2G}$ provided that $0 < |x-a| < \delta_1$ with $x \in D$.

Similarly, since $\lim_{x \to a} g(x) = B$, then $\exists \delta_2 > 0$ such that $|g(x) - B| < \frac{\varepsilon}{2|A|+1}$ provided that $0 < |x-a| < \delta_2$.

Let $\delta^* = \min\{\delta_1, \delta_2\}$. Then if $0 < |x-a| < \delta^*$, we have that $|f(x) - A| < \frac{\varepsilon}{2G}$ and $|g(x) - B| < \frac{\varepsilon}{2|A|+1}$

Then
$$\begin{aligned}|f(x)g(x) - AB| &= |f(x)g(x) - g(x)A + g(x)A - AB| \\ &\leq |g(x)||f(x) - A| + |A||g(x) - B| \\ &< (G)\left(\frac{\varepsilon}{2G}\right) + |A|\left(\frac{\varepsilon}{2|A|+1}\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therfore, with $0 < |x-a| < \delta^*$, we have that $|f(x)g(x) - AB| < \varepsilon$, so $\lim_{x \to a}[f(x) \cdot g(x)] = AB$.

Excellent!

7. State and prove the Bolzano-Weierstrass Theorem for Sets.

Any infinite and bounded subset of $\mathbb{R}$ has at least one accumulation point.

proof: Let S be an infinite and bounded subset of $\mathbb{R}$. Then it must have a lower and upper bound, which we will call a_1 and b_1. Then, the interval $[a_1, b_1]$ contains infinetly many points of S. Let $c_1 = \frac{a_1+b_1}{2}$. Then, either $[a_1, c_1]$ or $[c_1, b_1]$ contains infinetly many points of S. and we will denote this as $[a_2, b_2]$. Repeating this process n times, we get : $a_1 \le a_2 \le \cdots \le a_n \le c_n \le b_n \le \cdots \le b_2 \le b_1$.

Thus, the sequences $\{a_n\}$ and $\{b_n\}$ are monotone and bounded, so they must converge by the MCT. We will denote their limits as A and B.

Well, $0 \le b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}$, which we know goes to 0 as $n \to \infty$, therefore, we must have that $A = B$.

Now, to prove that A is an accumulation point, let $\varepsilon > 0$.

Then, for large enough values of n, we know that $A - \varepsilon < a_n \le b_n < A + \varepsilon$.

Since $[a_n, b_n]$ contains infinetly many points of S, we know that the interval $(A-\varepsilon, A+\varepsilon)$ does too. Therefore, there must be an element of S contained within this interval that is not equal to A.

Thus, A is an accumulation point. □

Excellent!

8. Any sequence for which $\lim_{n\to\infty} a_n = L$ for some real limit L must be bounded. Is the same true for a function f if you know $\lim_{x\to\infty} f(x) = L$? Why or why not?

While $\lim_{x\to\infty} f(x) = L$ implies that $f(x)$ is eventually bounded, convergent functions may not be bounded everywhere on their domain. Consider $f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & \text{else} \end{cases}$

This function converges to 0 as x approaches infinity, yet $f(x) \to \infty$ as x approaches 0 from the right...

I believe this is sufficient to say $f(x)$ is not bounded, because bounded $\iff |f(x)| < M \in \mathbb{R} \ \forall x \in$

Excellent!

9. Why is the requirement that a be an accumulation point of the domain D included in the definition of the limit of f as x approaches a?

If a is not an accumulation point of D, then x doesn't approach a, per se. There's a gap around a which is large enough to fit small δs, which would vacuously fulfill the definition for many limits.

i.e., there could be no $x \in D$ s.t. $|x-a| < \delta$, so "all" the xs that work (none) satisfy $|f(x)-L_1| < \varepsilon$ and $|f(x)-L_2| < \varepsilon$, with $L_1 \neq L_2$.

Exactly!

10. Show that for any $x \in \mathbb{R}^+$, there exists $n \in \mathbb{N}$ such that $n - 1 \leq x < n$.

Well, by the Archimedean Property we know there exists at least one natural number greater than x. But then the set of naturals greater than x is non-empty, so it follows from the Completeness Axiom that it has a least element. Call that element n. So we have $x < n$. It must also be that $n-1 \leq x$, or otherwise n would not be the least element of $\mathbb{N}$ greater than x. $\square$