Exam 2 **Real Analysis 1** **11/1/24**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the derivative of a function $f(x)$ at $x = a$.

Suppose $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, a is an accumulation point of D, and $a \in D$. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ is the derivative

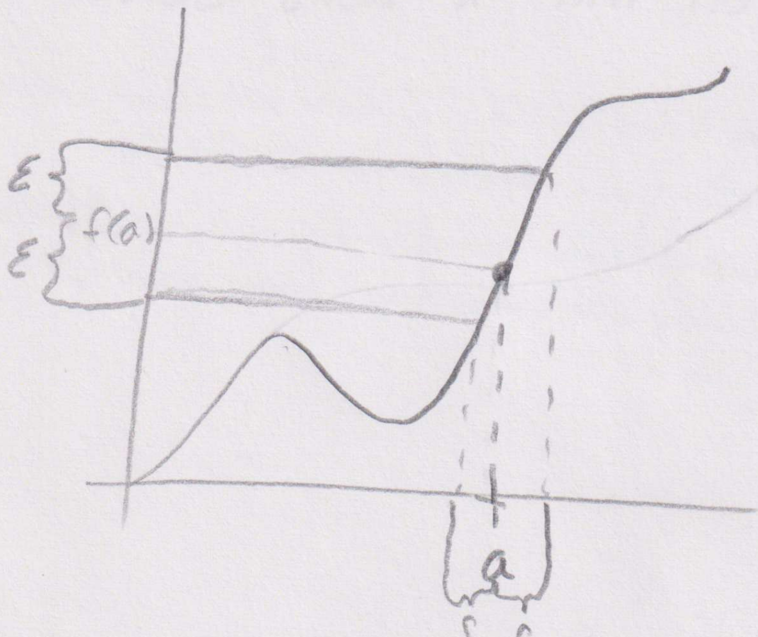
of $f(x)$ at $x = a$ provided the limit exists and is finite.

Excellent!

2. State the definition of a function f being continuous at $x = a$.

A function $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ is continuous at $x=a$ iff $\forall \varepsilon > 0\ \exists \delta > 0 \ni |x-a| < \delta$ and $x \in D \Rightarrow |f(x) - f(a)| < \varepsilon$

Great



3. a) State the Intermediate Value Theorem.

If f is continuous on $[a,b]$ and k is between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ for which $f(c) = k$.

b) State Brouwer's Fixed-Point Theorem

If $f : [a,b] \to [a,b]$ is continuous, then there exists a fixed point $x_0 \in (a,b)$ for which $f(x_0) = x_0$.

4. a) State Fermat's Theorem.

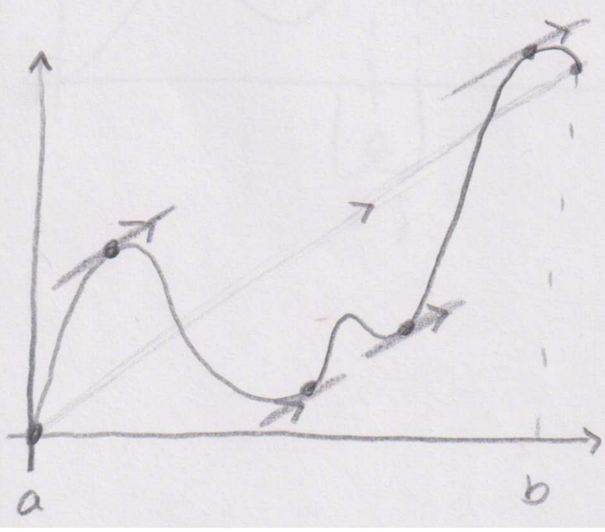
let $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ with a relative extremum at $c \in (a,b) \subseteq D$, differentiable at c, then $f'(c) = 0$.

Good

b) State the Mean Value Theorem.

If f is a continuous function on $[a,b]$ and f is differentiable on (a,b), then $\exists c \in D$ $\ni f'(c) = \frac{f(b)-f(a)}{b-a}$ given $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$

Great!



$\exists f'(c) = \frac{f(b)-f(a)}{b-a}$

5. a) State the definition of a compact set.

A set E is compact iff every open cover of E has a finite subcover.

b) State the Heine-Borel Theorem.

In $\mathbb{R}$, a set is compact iff it is closed and bounded.

c) Give an example of an open cover for $\mathbb{R}$ that has no finite subcover.

$\mathfrak{C} = \{(n, n+\frac{3}{2}) : n \in \mathbb{Z}\}$.

Yes!

6. State and prove the Extreme Value Theorem.

Thm: If f is continuous on $[a,b]$ then it attains its maximum and minimum on $[a,b]$

Pf By the Boundedness Thm., we know f is bounded above. By the Completeness Axiom we know f has a LUB, call it M. Suppose $\nexists c \in [a,b] \ni f(c) = M$, then $\forall x \in [a,b]$, $f(x) < M$. Now let $g(x) = \frac{1}{M - f(x)}$, noting that g is continuous on $[a,b]$. Thus by Boundedness Thm, g is bounded above by some K, so $\forall x \in [a,b]$ $g(x) < K$ or $\frac{1}{M - f(x)} < K$. Then, $M - f(x) < 1/K$ or $f(x) < M - 1/K < M$ contradicting M being our LUB.

Let $h(x) = -f(x)$ which is continuous on $[a,b]$. From above, we know that at some $d \in [a,b]$, $h(d) = U$ where U is a max of h on $[a,b]$. So, $-f(d) = U$ and $\forall x \in [a,b]$, $-f(x) \leq U$. Then, $f(d) = -U$ and $\forall x \in [a,b]$, $f(x) \geq -U$, so $-U$ is the GLB of f on $[a,b]$ and $\exists d \in [a,b] \ni f(d) = -U$.

Beautiful!

7. State and prove Rolle's Theorem.

Thm. If f is continuous on $[a,b]$
differentiable on (a,b)
and $f(a)=f(b)$
then $\exists\, c \in (a,b) \ni f'(c)=0$

Pf.
Case 1: $f(x)$ is a constant function.
Then, $\forall x \in (a,b)$, $f(x)=0$.

Case 2: For some $x \in (a,b)$, $f(x) > f(a)$.
Then, by EVT we know f attains its max at some $c \in [a,b]$ but we know $f(a)$ is not a max and therefore $f(b)$ is not a max so $c \in (a,b)$. Thus by Fermat's Theorem $f'(c)=0$

Case 3: For some $x \in (a,b)$ $f(x) < f(a)$.
By EVT we know f attains its min at some $d \in [a,b]$, but $f(a)$ is not a min and therefore $f(b)$ is not a min so $d \in (a,b)$. Thus by Fermat's Thm. $f'(d)=0$.

Great

8. Prove that the product of two functions that are continuous at $x = a$ is continuous at $x = a$.

let $f, g : D \to f$ be continuous at $x = a$.

Then, $\forall \varepsilon > 0, \exists \delta_g > 0 \ni |x - a| < \delta_g, x \in D \Rightarrow |g(x) - g(a)| < \frac{\varepsilon}{2|f(a)|+1}$

(basically boundedness thm) $\forall \varepsilon > 0, \exists \delta_K > 0 \ni x \in (a - \delta_K, a + \delta_K)\ x \in D \Rightarrow |g(x)| < K \in \mathbb{R}^+$

$\forall \varepsilon > 0, \exists \delta_f > 0 \ni |x - a| < \delta_f, x \in D \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2K}$

As such, $\forall \varepsilon > 0, \exists \delta^* = \min\{\delta_g, \delta_K, \delta_f\} \ni$

$|fg(x) - fg(a)| = |f(x)g(x) - f(a) - g(a)|$

$= |f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)| = |g(x)(f(x) - f(a)) + f(a)(g(x) - g(a))|$

(by triangle inequal.) $\leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)|$

by assumptions $< K|f(x) - f(a)| + |f(a)||g(x) - g(a)|$

by def^n continuity $< K\left(\frac{\varepsilon}{2K}\right) + |f(a)| \cdot \frac{\varepsilon}{2|f(a)|+1}$

because $\frac{a}{2b+1} < \frac{a}{2b}$ $< K\left(\frac{\varepsilon}{2K}\right) + |f(a)| \cdot \frac{\varepsilon}{2|f(a)|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So by def^n continuity, fg is continuous @ $x = a$ $\square$

Great!

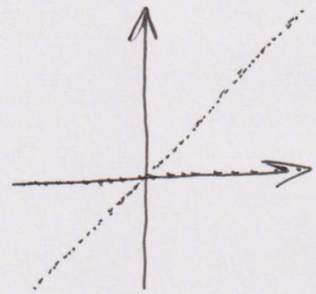
9. Let E be a set, with $E \subseteq \mathbb{R}$. Show that if E is closed, then $\mathbb{R} \setminus E$ is open.

Let E be a closed set. Then all the accumulation points of E are elements of E.

Suppose $\mathbb{R}-E$ were not open. then,
there exists some $a \in \mathbb{R}-E$ such that any neighborhood of a intersects E. That's the def$^{\underline{n}}$ of a being an accumulation point of E, but $\mathbb{R}-E \cap E = \emptyset$, and E is supposed to have all its accumulation points, so we have a contradiction.

Good.

10. Let $q(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Where is $x \cdot q(x)$ continuous? Where is it differentiable? Justify your answers well.



Claim 1: $x \cdot q(x)$ is not continuous when $x \neq 0$.

Proof: Well, take $x \neq 0$ and let $\varepsilon = \frac{|x|}{2}$. Then for any $\delta > 0$ we have both rationals and irrationals in $(x-\delta, x+\delta)$. If x is rational that makes an irrational a within δ of x, and $|f(x) - f(a)| = |x - 0| \not< \frac{|x|}{2}$. If instead x is irrational that means there is a rational b within δ of x for which $|f(x) - f(b)| = |0 - b| \not< \frac{|x|}{2}$. □

Claim 2: $x \cdot q(x)$ is continuous when $x = 0$.

Proof: Well, take $x = 0$ and let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Then for $|x - 0| < \delta$ we have either $x \cdot q(x) = 0$ or $x \cdot q(x) = x$, so $|x \cdot q(x) - 0| = |x - 0| < \delta = \varepsilon$. □

Claim 3: $x \cdot q(x)$ is non-differentiable everywhere.

Proof: Well, if $a \neq 0$ then the function is not continuous and thus not differentiable. If $a = 0$ then we're looking at $\frac{x \cdot q(x) - 0 \cdot q(0)}{x - 0}$, or just $\frac{x \cdot q(x)}{x} = q(x)$, and we already knew that this function has no limit anywhere. □