Examlet 4A Foundations of Advanced Math 4/19/13

1. a) State the definition of a reflexive relation.

A relation R on a set S is reflexive iff $\forall a \in S$, $(a,a) \in R$.

b) Give an example of a relation on the set $\{a, b, c\}$ which is transitive but not symmetric.

The relation $\{(c,b)\}$ is (vacuously) transitive, but not symmetric because it has (c,b) without having (b,c).

2. Let $S = \{1, 2, 3, 4, 5\}$, and consider the partition $\mathcal{P} = \{\{1,2\},\{3, 4, 5\}\}$ of S. Write the equivalence relation $\sim$ corresponding to $\mathcal{P}$.

$\sim = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),$
$(5,4),(5,5)\}$

Great

3. a) Express the definition of an injective function formally in terms of ordered pairs.

$$(x_1, a), (x_2, a) \in f \Rightarrow x_1 = x_2$$

Good

b) Express the definition of an even function formally in terms of ordered pairs.

$$(x, y) \in f \Rightarrow (-x, y) \in f.$$

Great

4. Let S be a set and $\mathcal{P}$ a partition of S.

a) The relation on S defined by $a \sim b$ iff $\exists P \in \mathcal{P}$ for which $a, b \in P$ is a reflexive relation.

If $a \in S$, there must be at least one set in $\mathcal{P}$ that contains a because the sets of the partition of S form S when unioned together. Since $\exists P \in \mathcal{P}$ for which $a \in P$, $a \in P$. Since $a, a \in P$, $a \sim a$. Therefore, this is a reflexive relation.

Nice

b) The relation on S defined by $a \sim b$ iff $\exists P \in \mathcal{P}$ for which $a, b \in P$ is a symmetric relation.

Suppose $a, b \in S$ and $a \sim b$, meaning that $\exists P \in \mathcal{P}$ for which $a, b \in P$. Since $a, b \in P$, $b, a \in P$. This means that $b \sim a$. Therefore, the relation is symmetric.

Great

c) The relation on S defined by $a \sim b$ iff $\exists P \in \mathcal{P}$ for which $a, b \in P$ is a transitive relation.

Suppose $a, b, c \in S$, $a \sim b$, and $b \sim c$. This means that $\exists P \in \mathcal{P}$ such that $a, b \in P$, and $\exists Q \in \mathcal{P}$ such that $b, c \in Q$. By our definition of partition, $\mathcal{P}$ is pairwise disjoint, meaning that two sets in $\mathcal{P}$ can only share elements if they are equal. Therefore if $b \in P$ and $b \in Q$, $P = Q$. Since $P = Q$ and $a \in P$, $a \in Q$. $a, c \in Q$, so $a \sim c$. Therefore, this is a transitive relation.

Excellent!

5. Say that two vertices v_1 and v_2 of a graph G are **propinquous** iff there exists a walk between them that contains exactly one vertex other than v_1 and v_2.

a) Is the relation of being propinquous reflexive?

Nope. In the graph $\bullet^{v_1}$ with no edges, there is no walk from v_1 to v_1 including another vertex.

b) Is the relation of being propinquous symmetric?

Yep. If $v_1 \sim v_2$ that means there exists a walk $v_1 e_1 v_3 e_2 v_2$ containing exactly one vertex other than v_1 and v_2. But if we reverse this, $v_2 e_2 v_3 e_1 v_1$ is a walk containing exactly one vertex other than v_1 and v_2, so $v_2 \sim v_1$.

c) Is the relation of being propinquous transitive?

Not in general. For example in 

there is a walk with exactly one other vertex from v_1 to v_2 and a walk with exactly one other vertex from v_2 to v_3, but no walk with exactly one other vertex from v_1 to v_3.

Note, however, that there are graphs like 

where any two vertices are joined by a walk containing exactly one other vertex, so on those graphs this relation will be reflexive.

Bonus