Problem Set 9        Real Analysis 1        Due 12/6/2002

Each problem is worth 5 points. Adequate demonstration is required for full credit.

1. Give an example of a function \( f: \mathbb{R} \to \mathbb{R} \) for which an upper sum on \([0,1]\) with four subdivisions gives three times the value of \( \int_0^1 f \, dx \).

Example: Well, there are many good candidates, most of them “spiky” shapes of one sort or another, but perhaps the simplest is to let \( f(x) = \frac{9}{4} \) when \( x = \frac{1}{8} \), and 1 everywhere else. \( \square \)

2. Give an example of a function \( f: \mathbb{R} \to \mathbb{R} \) for which a lower sum on \([0,1]\) with four subdivisions is zero even though the value of \( \int_0^1 f \, dx \) is not zero.

Example: Well, again there lots of ways to do this, but consider \( f(x) = \sin(8\pi x) + 1 \) for one clean example.

3. Prove or give a counterexample: If functions \( f \) and \( g \) are not Riemann integrable, then \( f+g \) is not Riemann integrable.

Counterexample: Well, let \( f(x) = 1 \) at any rational and 0 at any irrational, and \( g(x) = 0 \) at any rational and 1 at any irrational. Then \( f+g(x) = 1 \) everywhere, which is Riemann integrable although \( f \) and \( g \) are not. \( \square \)

4. Prove or give a counterexample: If \( f: \mathbb{R} \to \mathbb{R} \) is a differentiable function, then \( f \) is Riemann integrable on any interval \([a,b]\).

Proof: Well, if \( f \) is differentiable then it’s continuous by a theorem from a while back. And if \( f \) is continuous then it’s Riemann integrable by a theorem in chapter 6, so any differentiable function has to be integrable. \( \square \)

5. Prove or give a counterexample: If \( f: \mathbb{R} \to \mathbb{R} \) is a Riemann integrable function on any interval \([a,b]\), then \( f \) is differentiable.

Counterexample: Well, how about the example I used in problem 1. \( \square \)

6. Prove that a constant function \( f(x) = c \), for \( c \in \mathbb{R} \), is Riemann integrable on any interval \([a,b]\).

Proof: Well, take any partition of \([a,b]\). The upper sum on this partition will be \((b-a)c\), and the lower sum will also be \((b-a)c\), since in both cases on any subinterval the highest and lowest heights the function take on will be \( c \). Then the upper and lower sums are always equal, so the function is Riemann integrable. \( \square \)