To find the highest and lowest points on the ellipse:
\[ \nabla f = \lambda \nabla g + \mu \nabla h \]
The highest and lowest points will be on the \( z \)-axis so:
\[ f(x, y, z) = 2 \quad g(x, y, z) = x^2 + y^2 - 2 = 0 \quad h(x, y, z) = x^2 + y^2 + z^2 = 24 \]

\[ x : 0 = \lambda (2x) + \mu (1) \]
\[ y : 0 = \lambda (2y) + \mu (1) \]
\[ z : 0 = \lambda (2y) + \mu (1) \]

\[ -\lambda = 2x \]
\[ -\mu = 2y \]
\[ 2x = 2y \]
\[ 2z = 2z \]
\[ \lambda = x \]

To find values for \( x, y, \) and \( z \):
\[ x^2 + y^2 - 2 = 0 \]
\[ 2x^2 - 2 = 0 \]
\[ 2x^2 - 24 - 2x = 0 \]
\[ x^2 + x = 12 \]
\[ x^2 + x - 12 = 0 \]
\[ (x+4)(x-3) \]
\[ x = -4, 3 \]

\[ y = x \]
\[ y = -4, 3 \]
\[ z = \text{varies} \]

\[ (-4, -4, 32) \quad \text{and} \quad (3, 3, 18) \]

The minimum point is \((3, 3, 18)\) and the maximum point is \((-4, -4, 32)\).

We know which point is the min and which is the max by looking at the \( z \) values. The largest \( z \) is the max and the smallest \( z \) is the min.
2. \[ Z = ax^2 - axy^2 \text{ paraboloid} \]
\[ Z = ax^2 \]
\[ V_p = \text{volume of frustum} \]
\[ V_c = \text{volume of approx. cylinder} \]

\[ V_p = \int_0^{2\pi} \int_0^{\arctan\left(\frac{a}{b}\right)} r^2 \cos^2 \theta \, d\theta \, dr - \int_0^{2\pi} \int_0^{\arctan\left(\frac{a}{c}\right)} r^2 \cos^2 \theta \, d\theta \, dr \]
\[ = \int_0^{2\pi} \frac{1}{4} b^4 \, d\theta - \int_0^{2\pi} \frac{1}{4} c^4 a^4 \, d\theta \]

\[ V_p = \frac{c\pi}{2} \left[ b^4 - a^4 \right] \]

\[ d = ca^2; e = cb^2 \]
\[ ca^2 = \frac{(d+e)}{2} = \frac{c[a^2 + b^2]}{2} \]
\[ r^2 = \frac{a^2 + b^2}{2} \]

\[ V_c = \left[ \frac{\text{Area}(r_1^2 \pi)}{2} \right] \left[ \text{Height} \right] \]

\[ V_c = \frac{\pi}{2} \left( a^2 + b^2 \right) \left( c^2 - a^2 \right) \]

\[ V_c = \frac{c\pi}{2} \left[ a^4 - b^4 \right] \]

Good
Problem 3

Euler also showed that the difference between a frustum of a right circular cone and the corresponding cylinder is one-fourth the volume of a similar cone, with the same height as the frustum and with diameter one-half the difference between the upper and lower diameters of the frustum. Use a double integral to express the volume of a frustum of a right circular cone and show why this is true.

**Volume of frustum:**
\[
\int_0^{2\pi} \int_0^{b/k} (b-kr) r \, dr \, d\theta - \int_0^{2\pi} \int_0^{a/k} (a-kr) r \, dr \, d\theta = \int_0^{2\pi} \int_0^{b/k} (br - r^2k) \, dr \, d\theta - \int_0^{2\pi} \int_0^{a/k} (ar - r^2k) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{b^3}{2k^2} - \frac{b^3}{3k^2} \right] \, d\theta - \int_0^{2\pi} \left[ \frac{a^3}{2k^2} - \frac{a^3}{3k^2} \right] \, d\theta = \int_0^{2\pi} \left[ \frac{b^3}{2k^2} - \frac{b^3}{3k^2} \right] \, d\theta - \int_0^{2\pi} \left[ \frac{a^3}{2k^2} - \frac{a^3}{3k^2} \right] \, d\theta = \frac{b^3\pi}{3k^2} - \frac{a^3\pi}{3k^2} = \frac{\pi}{3k^2} \left( b^3 - a^3 \right) = V_f
\]

**Volume of approximating cylinders:**
\[
\pi (b-a) \left( \frac{b+a}{2k} \right)^2 = \pi (b-a) \left( \frac{b^2 + ab + a^2}{4k^2} \right) = \frac{\pi}{4k^2} \left( b^3 + ab^2 + a^2b - ab^2 - a^2b - a^3 \right)
\]
\[
\frac{\pi}{4k^2} \left( b^3 + ab^2 - a^2b - a^3 \right) = V_c
\]
Volume of the相似 cone:

\[ V = \frac{1}{3} \pi \left( \frac{b-a}{k} \right)^2 (b-a) = \frac{\pi}{3k^2} \left( b^2 - dab + a^2 \right) \left( b - a \right) = \]

\[ \frac{\pi}{3k^2} \left( b^3 - 3ab^2 + 3a^2 b - a^3 \right) = V_\Delta \]

Determine the difference between the approximating cylinder and frustum:

\[ D = \frac{\pi}{3k^2} \left( b^3 - a^3 \right) - \frac{\pi}{4k^2} \left( b^3 + ab^2 - a^2 b - a^3 \right) \]

\[ = \frac{4\pi}{12k^2} \left( b^3 - a^3 \right) - \frac{3\pi}{12k^2} \left( b^3 + ab^2 - a^2 b - a^3 \right) \]

\[ = \frac{\pi}{12k^2} \left( b^3 - 3ab^2 + 3a^2 b - a^3 \right) = V_{D, \text{diff}} \]

\[ \therefore \quad \frac{1}{4} (V_\Delta) = V_{D, \text{diff}} \]

Excellent
4) a - see graph

b - this temperature distribution is likely to be caused by a wall heater at one end of a room.

c - Mathematica evaluates
\[ \int_{-3}^{3} \int_{0}^{5} (6x - 6x^2 + 2x^3 - 0.002x^4) \, dy \, dx \]
to be 2173.72. Dividing this by the area of the room (5.6 x 30) we get:

\[ \frac{2173.72}{30} = 72.4573 \approx 72.5 \]

which is the average height (or temperature in degrees centigrade) for the function over the range specified.

Great
5. a) \( x^{2/3} + y^{2/3} + z^{2/3} = 1 \). Solve for \( z \): \( z = (1 - x^{2/3} - y^{2/3})^{3/2} \) & graph:

\[
\int_0^1 \int_0^{(1-x^{2/3})^{3/2}} (1 - x^{2/3} - y^{2/3})^{3/2} \, dy \, dx
\]

Solve for \( x \) or \( y \) when \( z = 0 = (1 - x^{2/3} - y^{2/3})^{3/2} \) yields \( x = (1 - y^{2/3})^{3/2} \) & \( y = (1 - x^{2/3})^{3/2} \).

A top down view of the first octant is shown. If you take the \( x \) values ranging from 0 to 1 and the \( y \) values ranging from 0 to \((1 - x^{2/3})^{3/2}\) then you can set up the double integral as follows: (8 times this value is the volume.)
c) By letting \( y = \sin^3 \theta (1 - x^{2/3})^{3/2} \) and then finding \( dy = 3\sin^2 \theta \cos \theta (1 - x^{2/3})^{3/2} \ d\theta \) and getting new limits of integration, you have the following integral which can be solved:

when \( y = 0, \ \theta = k\pi = 0 \) (if \( k = 0 \))
when \( y = (1 - x^{2/3})^{3/2}, \ \theta = 2k\pi + \pi/2 = \pi/2 \)

\[
\int_{\frac{\pi}{2}}^{\pi} \int_{0}^{x^{2/3}} \left[ 1 - \left( \sin^3 \theta \left( 1 - x^{2/3} \right)^{3/2} \right)^{3/2} \left( 1 - x^{2/3} \right)^{3/2} \right] 3\sin^2 \theta \cos \theta \ d\theta \ dx
\]

\[
= \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{x^{2/3}} \left( 1 - \sin^2 \theta \right)^{3/2} 3\sin^2 \theta \cos \theta \ d\theta \ dx = \frac{\pi}{70}
\]

and since this is the volume of one octant, the resulting volume for the entire solid is: \( \frac{8\pi}{70} = \frac{4\pi}{35} \approx 0.359 \)